PERMUTATION WEIGHTS FOR $A_r^{(1)}$ LIE ALGEBRAS

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Abstract

When it is based on Kac-Peterson form of Affine Weyl Groups, Weyl-Kac character formula could be formulated in terms of Theta functions and a sum over finite Weyl groups. We, instead, give a reformulation in terms of Schur functions which are determined by the so-called **Permutation Weights** and there are only a finite number of permutation weights at each and every order of weight depths .

Affine signatures are expressed in terms of an index which by definition is based on a decomposition of horizontal weights in terms of some **Fundamental Weights**.

I. INTRODUCTION

Although we will consider only $A_r^{(1)}$ algebras in this work, this section is valid in general for any other affine Lie algebra for which we follow the notation of Kac [**1**].

Just as in Weyl character formula [**2**], in any application of Weyl-Kac character formula the central idea is again to calculate

$$A(\Lambda) \equiv \sum_{\omega} \epsilon(\omega) \ e^{\omega(\Lambda)} \qquad (I.1)$$

where ω is an element of affine Weyl group and $\epsilon(\omega)$ is the corresponding signature. The formal exponentials $e^{\omega(\Lambda)}$ are defined as in the book of Kac [**1**]. This is, however, not our point of view all along this work. Let $\tilde{\rho}$ and Λ^+ be, respectively, affine Weyl vector and an affine dominant weight. Then, we know [**3**] that (I.1) is equivalent to

$$A(\tilde{\rho} + \Lambda^+) \equiv \sum_{\Lambda \in W(\tilde{\rho}+\Lambda^+)} \epsilon(\Lambda) \ e^{\Lambda} \qquad (I.2)$$

where $W(\tilde{\rho} + \Lambda^+)$ is the corresponding Weyl orbit. This, hence, forces us to know the complete weight structure of Weyl orbits of affine dominant weights. To this end, two main characters are two isotropic elements Λ_0 and δ which are defined by

$$(\Lambda_0, \Lambda_0) = 0 \quad , \quad (\delta, \delta) = 0 \quad , \quad (\delta, \Lambda_0) = 1 \qquad (I.3)$$

where the existence of a symmetric scalar product are always known to be valid. In fact, one gets affine $A_r^{(1)}$ Lie algebra when a finite A_r Lie algebra is horizontally coupled to system (I.3). Horizontally coupled here means that

$$(\delta, \mu) = (\Lambda_0, \mu) = 0 \qquad (I.4)$$

where μ is an element of A_r Weight Lattice. Any element Λ of $A_r^{(1)}$ Weight Lattice can then be expressed in the general form

$$\Lambda = k \ \delta - M \ \Lambda_0 + \mu \qquad (I.5)$$

where k is the **level** and M the **depth** of Λ. In (I.5), the level and depth are defined to be non-negative integers. Note here that the depth is always assumed to be zero for an affine dominant weight Λ^+ within an integrable irreducible representation $R(\Lambda^+)$. As in (I.5), a similar expression

$$\sigma(\Lambda^+) = k \ \delta - M_\sigma \ \Lambda_0 + \mu_\sigma \qquad (I.6)$$

is valid for any element $\sigma(\Lambda^+)$ of affine Weyl orbit $W(\Lambda^+)$ where σ is an affine Weyl reflection, that is an element of affine Weyl group $W(A_r^{(1)})$. Note also that the level is always the same for all weights of $R(\Lambda^+)$ and that μ_σ is again horizontal. A definitive condition here is known to be

$$(\sigma(\Lambda^+), \sigma(\Lambda^+)) = (\Lambda^+, \Lambda^+) \quad . \tag{I.7}$$

(I.7) is, however, insufficient in many cases one of which is the case we studied here. At this point, the so-called permutation weights help us to determine all the weights participating in the same affine Weyl orbit completely. This will be clarified in section.III in which we give a constructive lemma for permutation weights. In view of this lemma, it will be seen that for the same affine Weyl orbit, there are always a finite number of permutation weights with the same depth and these permutation weights determine the contributions to Weyl-Kac formula. This, in fact, reflects our main point of view here in this work.

To be more specific, if one express the right-hand side of Weyl-Kac formula by

$$\sum_{M}^{\infty} C(M)\; q^M \quad ,$$

contributions to $C(M_0)$ for any fixed value M_0 could come only from permutation weights having the same value M_0. This means that we can control the contributions to Weyl-Kac character formula at each and every finite order M individually. It is clear that this is contrary to the one's expectation because in applications, which are based on Kac-Peterson form of affine Weyl groups, there is a summation over a part of the whole root lattice and hence, for instance, two different roots with the same length could in general contribute in more than one C(M). This will also be clarified by an example in section.IV.

II. CALCULATION OF A_r CHARACTERS

It will be useful to give here a brief discussion of how we consider A_r Lie algebras. This provides us a comparative look on results we obtained for both affine $A_r^{(1)}$ and its finite horizontal algebra A_r. For finite Lie algebras, we refer to the excellent book of Humphreys [**4**].

Let, for $i = 1, 2 \ldots r$, α_i's and λ_i's be simple roots and fundamental dominant weights of A_r, respectively. Our essential point of view is again to use **fundamental weights** μ_I

$(I = 1, 2 \ldots r+1)$ as building blocks. We define the system $\{\alpha_i\}$ of simple roots

$$\alpha_i \equiv \mu_i - \mu_{i+1} \tag{II.1}$$

as being in line with the following A_r Dynkin diagram:

Together with the condition

$$\sum_{I=0}^{r+1} \mu_I \equiv 0 \quad , \tag{II.2}$$

(II.1) gives us a complete definition of our fundamental weights. For only A_r Lie algebras fundamental weights become the weights of (r+1)-dimensional fundamental representation $R(\lambda_1)$. This is however not the case for any other finite Lie algebra.

For $s = 1, 2, \ldots r$, any dominant weight λ^+ then has the form

$$\lambda^+ = q_1 \ \mu_1 + q_2 \ \mu_2 + \ldots + q_s \ \mu_s \tag{II.3}$$

where

$$q_1 \geq q_2 \geq \ldots \geq q_s > 0 \quad . \tag{II.4}$$

(II.3) and (II.4), show that there is always a unique partition

$$\{\lambda^+\} \equiv \{q_1, q_2 \ldots q_s\} \tag{II.5}$$

which can be defined naturally for any A_r dominant weight λ^+. It is, in fact, nothing but a partition of

$$h(\lambda^+) \equiv q_1 + q_2 + \ldots + q_s \tag{II.6}$$

which is known to be the **height** of λ^+. Then, for any partition (II.5), one can define a Schur function

$$S_{\{\lambda^+\}}(x_1, x_2, .., x_r) = Det \begin{pmatrix} S_{q_1-0} & S_{q_1+1} & S_{q_1+2} & \cdots & S_{q_1+s-1} \\ S_{q_2-1} & S_{q_2+0} & S_{q_2+1} & \cdots & S_{q_2+s-2} \\ S_{q_3-2} & S_{q_3-1} & S_{q_3+0} & \cdots & S_{q_3+s-3} \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix} \tag{II.7}$$

where Det means determinant and x_i's are some indeterminates given below. For convenience, we keep here the notation $S_q \equiv S_{\{q \ \lambda_1\}}(x_1, x_2, \ldots, x_r)$.

When one considers, on the other hand, (I.1) for A_r Lie algebras, it is known that

$$A(\rho+\Lambda^+) = Det \begin{pmatrix} u_1^{q_1+r} & u_1^{q_2+r-1} & \dots & u_1^{q_{r+1}} \\ u_2^{q_1+r} & u_2^{q_2+r-1} & \dots & u_2^{q_{r+1}} \\ \vdots & \vdots & \vdots & \vdots \\ u_{r+1}^{q_1+r} & u_{r+1}^{q_2+r-1} & \dots & u_{r+1}^{q_{r+1}} \end{pmatrix} \qquad (II.8)$$

in the specialization

$$e^{\mu_I} = u_I \quad , \quad I = 1,2\dots r+1 \qquad (II.9)$$

of formal exponentials. With the aid of substitution

$$x_i \equiv \frac{u_1^i + u_2^i + \dots + u_{r+1}^i}{i} \qquad (II.10)$$

in (II.8), one can show that

$$A(\rho+\lambda^+) = A(\rho)\ S_{\{\lambda^+\}}(x_1,x_2\dots x_r) \quad . \qquad (II.11)$$

This last expression is of central importance in this work and for a discussion of how one reaches (II.11) we refer to a previous work **[5]** in which we also give a very detailed study of an A_5 example.

III. A CALCULATION OF $A_r^{(1)}$ CHARACTERS

With the inclusion of an extra simple root α_0 defined by

$$\alpha_0 \equiv \mu_{r+1} - \mu_1 + \delta \qquad (III.1)$$

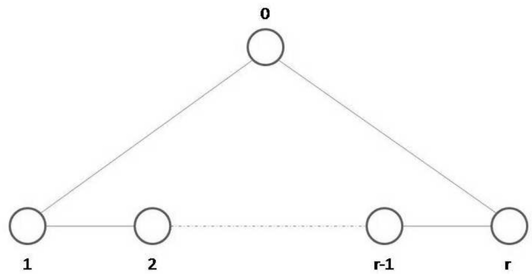
one gets affine $A_r^{(1)}$ in view of those given in previous two sections. The crucial point here is to see, from (I.3), that α_0 is dual to Λ_0. This, hence, allows us to obtain an affine system

$$\{\alpha_\nu \quad , \quad \nu = 0,1,2\dots r\}$$

of simple roots for $A_r^{(1)}$ whereas, in addition to Λ_0, a dual system of affine fundamental dominant weights Λ_ν's can then be defined by

$$\Lambda_i = \Lambda_0 + \lambda_i \quad , \quad i = 1,2\dots r \qquad (III.2)$$

with respect to the following extended Dynkin diagram:



Any other affine dominant weight Λ^+ of level k then has the form

$$\Lambda^+ = k\ \Lambda_0 + \lambda^+ \qquad (III.3)$$

where λ^+ is an A_r dominant weight with $h(\lambda^+) \leq k$. It is known **[1]** that there is an integrable highest weight representation of $A_r^{(1)}$ for each and every affine dominant weight of the form (III.3). Note here that adjoint representation is not of this type because it has k=0.

As we addressed in section I, (I.1) leaves us together with two main problems one of which is a complete description of weights participating in the same affine Weyl orbit. Note here that the problem arises due to the fact that the condition (I.5), which expresses the invariance of weight lenths under affine Weyl reflections, becomes sufficient only for affine dominants with $k = 1$ while it remains insufficient in general for a dominant weight with $k > 1$. It is this fact which leads us to state the following lemma which manifests why we need to define permutation weights.

<u>DEFINITION:</u> Let $\Lambda^+ = k\ \Lambda_0 + \lambda^+$ be $A_r^{(1)}$, μ^+ and λ^+ A_r dominant weights. Then, $\sigma(\Lambda^+)$ is called a **permutation weight** for $W(\Lambda^+)$ if there is an element $\sigma \in W(A_r^{(1)})$ such that

$$\sigma(\Lambda^+) = k\ \Lambda_0 - M\ \delta + \mu^+ \ . \qquad (III.4)$$

It will be useful to recall here that, as all other elements of the same Weyl orbit, any permutation weight solves the equation

$$(\mu^+, \mu^+) - (\lambda^+, \lambda^+) = 2\ k\ M \ . \qquad (III.5)$$

Let us also remark here that by saying an A_r dominant weight μ^+ solves (III.5) we mean there is a corresponding permutation weight as in (III.4). This will be useful in the

understanding of the lemma given below. To obtain a complete list of permutation weights we have now two steps one of which, the corollary, is for only affine fundamental dominant weights Λ_ν.

COROLLARY: Let, for an affine dominant weight Λ^+, $\mathcal{P}(\Lambda^+, M)$ be the set of permutation weights with depths $\leq$ M. Then,

(1) any solution of (III.5) is in $\mathcal{P}(\Lambda_\nu, M)$,

(2) $\mathcal{P}(\Lambda_\nu, M)$ is finite dimensional for finite M .

The second part of this corollary can be easily seen by noting that one always has

$$\mu^+ = \lambda^+ + \sum_{i=1}^{r} n_i \ \alpha_i$$

where n_i's are some non-negative integers. It is clear that only a finite number of sets $\{n_1, n_2 \ldots n_r\}$ can solve (III.5) and hence, at each and every order M of weight depths, there is only a finite number of contributions to Weyl-Kac formula.

For any other affine dominant weight (III.3), the problem of a complete determination of the corresponding $\mathcal{P}(\Lambda^+, M)$ will then be provided by the following lemma.

LEMMA: For an affine dominant $\Lambda^+ = \Lambda_{\nu_1} + \Lambda_{\nu_2} + \ldots + \Lambda_{\nu_n}$, any μ^+ which solves (III.5) is in $\mathcal{P}(\Lambda^+, M)$ on condition that

$$\mu^+ \equiv \theta_1 + \theta_2 + \ldots + \theta_n \qquad (III.6)$$

where $\theta_1 \in \mathcal{P}(\Lambda_{\nu_1}, M_1)$, $\theta_2 \in \mathcal{P}(\Lambda_{\nu_2}, M_2)$, ... , $\theta_n \in \mathcal{P}(\Lambda_{\nu_n}, M_n)$ and

$$M = M_1 + M_2 + \ldots + M_n \quad .$$

providing $M \geq M_j$ for $j = 1, 2, \ldots n$.

The second problem mentioned at the beginning of this section is to determine signatures $\epsilon(\sigma)$ for an infinite number of affine Weyl reflection σ. We instead find useful to solve this problem by defining a new index which gives us the precise values of signatures $\epsilon(\Lambda)$ for any $\Lambda \in W(\rho + \Lambda^+)$. The fundamental weights μ_I will play again a major role in the construction of such an index for signatures.

DEFINITION: Let $\epsilon(s_1, s_2, \ldots, s_{r+1})$ be completely antisymmetric with the normalization

$$\epsilon(s_1, s_2, \ldots, s_{r+1}) = +1 \quad , \quad s_1 \geq s_2 \geq \ldots \geq s_{r+1} \quad . \qquad (III.7)$$

Let n_I's be non-negative integers and s_I's be some integers taking their values from the finite set $\{0,1,2\dots k\}$. Let also define

$$\epsilon \quad : \quad \sum_{I=1}^{r+1} (s_I + k\ n_I)\ \mu_I \rightarrow \epsilon(s_1, s_2, \dots, s_{r+1}) \quad . \tag{III.8}$$

Note here that ϵ is a mapping from A_r weight lattice to $Z_2 \equiv \{+1,-1\}$. In view of the fact that $h(\mu^+) \leq k$, it is obvious that any permutation weight μ^+ has a decomposition of the form

$$\mu^+ = \sum_{I=1}^{r+1} (s_I + k\ n_I)\ \mu_I \quad . \tag{III.9}$$

Its signature $\epsilon(\mu^+)$ then will be equal to the value of the mapping (III.8) on the point μ^+ of A_r Weight lattice.

All these, when they are considered together with those in ref.[**5**] concerning A_r Lie algebras, give us the possibility to express (I.1) in a form which is our ultimate result here in this work:

$$A(\Lambda^{++}, M) \quad \underset{M \rightarrow \infty}{\longrightarrow} \quad A(\Lambda^{++}) \tag{III.10}$$

where

$$A(\Lambda^{++}, M) \equiv A(\rho) \sum_{\mu^+ \in \mathcal{P}(\Lambda^{++}, M)} S_{\{\mu^+ - \rho\}}(x_1, x_2 \dots x_{r+1}) \tag{III.11}$$

It is therefore clear that, for any chosen value M_0 of M, (III.11) determines the contributions to Weyl-Kac formula up to order $M_0 + 1$. An example will be given in the next section.

IV. FROM TETA FUNCTIONS TO SCHUR FUNCTIONS

Our main exposition in this work is to express affine characters in terms of Schur functions just as in case of finite Lie algebras. This, on the other hand, could seem to be contrary to the one's expectation because in the conventional scheme affine characters are expressed in terms of teta functions. In the light of the work of Kac and Peterson [**6**], this is natural because affine Weyl groups include some kind of translations. Beyond that, another characteristic difference between the two approaches comes out in actual computations of affine characters. To this end, a brief discussion of the interplay between

Weyl-Kac formula and teta functions will be useful. The discussion below considered only for $A_r^{(1)}$ Lie algebras though it could be easily generalized for any affine Lie algebra.

In its classical treatment, the Jacobi's teta function is known to be defined by

$$\theta_3(\tau,z) \equiv \sum_{n=-\infty}^{\infty} e^{\pi i \tau n^2} \ e^{2\pi i z n} \tag{IV.1}$$

which allows us to write

$$\theta_3(\tau,\frac{s}{r+1}) \equiv 1 \ + \ \sum_{n=1}^{\infty} (u(sn)+u(-sn)) \ q^{\frac{1}{2}n^2}$$

where $q \equiv e^{2\pi i \tau}$ and formal exponentials

$$u(s) \equiv e(\frac{2\pi i}{r+1}s) \quad , \quad s = \mp 1, \mp 2, \ldots$$

can be defined in such a way that the limits $u(s) \rightarrow 0$ is valid formally for all values of integers s, positive or negative. One easily shows then that the lattice function $\Theta(A_r)$ of root lattice $\Gamma(A_r)$ is expressed, for $i = 1, 2, \ldots r$, by

$$\theta_3(\tau,\tfrac{i}{r+1})^{r+1} \quad \underset{u(s)\rightarrow 0}{\longrightarrow} \quad \Theta(A_r) \tag{IV.2}$$

Let us remark here that the more usual formulation of $\Theta(A_r)$ can be found in the book of Conway and Sloane [**7**]. Now, if one recalls

$$\Gamma(A_r) = \bigcup_{n=0}^{\infty} \Gamma_n$$

where

$$\Gamma_n \equiv \{\alpha \mid (\alpha,\alpha) = 2n\} \quad , \tag{IV.3}$$

the definition

$$\Theta(A_r) \equiv \sum_{n=0}^{\infty} \ q^n \ dim\Gamma_n \tag{IV.4}$$

is valid. One has, for instance,

$$\Theta(A_4) = 1 + 20 \ q + 30 \ q^2 + 60 \ q^3 + 60 \ q^4 + 120 \ q^5 + 40 \ q^6 + 180 \ q^7 + 150 \ q^8 + \ldots \tag{IV.5}$$

For $A_r^{(1)}$ Lie algebras, on the other hand, a natural extension

$$\theta_3(\tau,z) \longrightarrow \theta_\Lambda(\tau,z_1,z_2,\ldots,z_r) \equiv \theta_\Lambda \tag{IV.6}$$

is valid for any affine weight Λ and some coordinates z_i. Let us define

$$\theta_\Lambda \equiv \sum_{\alpha \in \Gamma(A_r)} q^{\frac{1}{2k}(\Lambda+k\alpha,\Lambda+k\alpha)} \prod_{i=1}^{r} u_i^{(\Lambda+k\alpha,\alpha_i)} \qquad (IV.7)$$

where $u_i \equiv e^{2\pi i z_i}$ and

$$k \equiv (\Lambda, \delta) \quad . \qquad (IV.8)$$

It is clear that (IV.7) is a kind of teta function. Its convenience, however, more clearly understood when one recalls that there is an appropriate spcecialization of Weyl-Kac character formula so that affine characters $Ch(\Lambda^+)$ are expressed by the aid of

$$B(\Lambda^+) \equiv \sum_{\omega \in W(A_r)} \epsilon(\omega)\ \theta_{\omega(\Lambda^+ + \tilde{\rho})} \quad . \qquad (IV.9)$$

What is remarkable in (IV.9) is that the sum is over the finite Weyl group $W(A_r)$. This is a result due to Kac-Peterson. (IV.9) can be cast into the form

$$B(\Lambda^+) \equiv \sum_{\alpha \in \Gamma_n} B(n, \Lambda^+) \qquad (IV.10)$$

where $B(n,\Lambda^+)$ can be extracted from definition (IV.7). An equivalent form can be given, however, as in the following:

$$B(n, \Lambda^+) = q^{a(A_r,\tilde{k})}\ B(0, 0^+) \quad T_{n,\Lambda^+}(q, u_1, u_2, \ldots, u_r) \qquad (IV.11)$$

where $T_{n,\Lambda^+}(q, u_1, u_2, \ldots, u_r)$ is a polinomial with integer powers of indeterminate q and $a(A_r, \tilde{k})$ is the so-called anomaly which is known to be defined by

$$a(A_r, \tilde{k}) \equiv \frac{1}{2\ \tilde{k}} \left((\Lambda^+, \Lambda^+ + 2\ \tilde{\rho}) - \frac{1}{12}\ \tilde{k}\ dim A_r \right) \qquad (IV.12)$$

where $\tilde{k} \equiv (\Lambda^+ + \tilde{\rho}, \delta)$ and $dim A_r = r(r+2)$. It is therefore seen that when one attempts to obtain affine characters the main problem is to calculate polinomials T_{n,Λ^+} . Weyl-Kac formula then gives us **normalized characters**

$$\chi_{\Lambda^+}(\tau, z_1, z_2, \ldots, z_r) = \frac{1 + T_{1,\Lambda^+} + T_{2,\Lambda^+} + \ldots}{1 + T_{1,0^+} + T_{2,0^+} + \ldots} \quad . \qquad (IV.13)$$

(IV.13) never permits to a complete factorization which means in effect that it consists of an infinite sum of powers of q. Although in a few simple cases first few elements lead us to compact expressions **[9]**, normalized characters can be given only up to some order

$N < \infty$, in general. It is clear that this brings out an unexpected problem in the theory because, as we stressed above, the polinomials T_{n,Λ^+} contain in general more than one terms with different powers of q. Let us first proceed with an example to illustrate the problem though we will give below how we consider the problem with the introduction of permutation weights.

One of the specialties of Affine Lie Algebras is the occurrence of basic representation $R(\Lambda_0)$ having the normalized character [**8**]

$$\chi_{\Lambda_0}(\tau) = \frac{\Theta(A_r)}{\Phi(\tau)^r} \qquad (IV.14)$$

where

$$\Phi(\tau) \equiv \prod_{n=1}^{\infty} (1 - q^n) \quad .$$

For convenience, we adopt the notation

$$\chi_{\Lambda_0}(\tau) = \chi_{\Lambda_0}(\tau, z_1 = 1, z_2 = 1, \ldots z_r = 1)$$

in the specialization used in (IV.14).

To be more abstract, let us proceed with A_4 for which following contributions to (IV.13) are obtained:

$$\begin{aligned}
T_{1,0^+} &= -\,24\ q + 252\ q^2 - 1472\ q^3 + 3654\ q^4 - \\
&\quad 19096\ q^6 + 40128\ q^7 - 34398\ q^8 + 10976\ q^9 \quad , \\
T_{2,0^+} &= 1176\ q^4 - 6048\ q^5 + 2352\ q^6 + 44352\ q^7 - 83997\ q^8 - \\
&\quad 78848\ q^9 + 360756\ q^{10} - 157248\ q^{11} - 530222\ q^{12} + \\
&\quad 598752\ q^{13} + 123552\ q^{14} - 448448\ q^{15} + 173901\ q^{16} \quad , \\
T_{1,\Lambda_0} &= -\,200\ q^2 + 2100\ q^3 - 9625\ q^4 + 19096\ q^5 - \\
&\quad 70200\ q^7 + 128625\ q^8 - 98000\ q^9 + 28224\ q^{10} \quad , \\
T_{2,\Lambda_0} &= 9000\ q^6 - 46200\ q^7 + 23100\ q^8 + 286000\ q^9 - 530222\ q^{10} - \\
&\quad 409500\ q^{11} + 1851850\ q^{12} - 754600\ q^{13} - 2281500\ q^{14} + \\
&\quad 2354352\ q^{15} + 477750\ q^{16} - 1528800\ q^{17} + 548800\ q^{18} \quad .
\end{aligned} \qquad (IV.15)$$

One sees now that

$$\frac{1 + T_{1,\Lambda_0}}{1 + T_{1,0^+}} = 1 + 24\ q + 124\ q^2 + \ldots \qquad (IV.16)$$

and

$$\frac{1+T_{1,\Lambda_0}+T_{2,\Lambda_0}}{1+T_{1,0^+}+T_{2,0^+}} = 1+24\ q+124\ q^2+ \\ 500\ q^3+1625\ q^4+4752\ q^5+12524\ q^6+31000\ q^7+\dots \tag{IV.17}$$

reproduces (IV.14) up to third and eigtheeth degree correctly. Both are insufficient to reproduce a correct q^8 term though all the polinomials in (IV.15) have non-zero q^8 terms. We need, in fact, third order contributions to calculate χ_{Λ_0} beyond q^7.

We will now show that we only need permutation weights $\mathcal{P}(\Lambda^+,2)$ and $\mathcal{P}(\Lambda^+,7)$ in order to calculate (IV.16) and (IV.17) respectively. In view of the lemma given above one indeed has

$$\begin{aligned}\mathcal{P}(\tilde\rho,8)=\{&(0,0,0,0)_0\ ,\ (1,0,0,1)_1\ ,\ (2,0,1,0)_2\ ,\ (0,1,0,2)_2\ ,\ (3,1,0,0)_3\ ,\\ &(1,1,1,1)_3\ ,\ (0,0,1,3)_3\ ,\ (5,0,0,0)_4\ ,\ (2,2,0,1)_4\ ,\ (1,0,2,2)_4\ ,\\ &(0,2,2,0)_4\ ,\ (0,0,0,5)_4\ ,\ (1,3,1,0)_5\ ,\ (0,1,3,1)_5\ ,\ (1,0,0,6)_6\ ,\\ &(0,5,0,0)_6\ ,\ (2,0,2,3)_6\ ,\ (3,2,0,2)_6\ ,\ (6,0,0,1)_6\ ,\ (0,0,5,0)_6\ ,\\ &(2,0,1,5)_7\ ,\ (3,1,1,3)_7\ ,\ (1,1,3,2)_7\ ,\ (5,1,0,2)_7\ ,\ (2,3,1,1)_7\ ,\\ &(3,1,0,5)_8\ ,\ (1,5,0,1)_8\ ,\ (0,1,0,7)_8\ ,\ (5,0,1,3)_8\ ,\ (2,2,2,2)_8\ ,\\ &(1,0,5,1)_8\ ,\ (7,0,1,0)_8\}\end{aligned}$$

$$\begin{aligned}\mathcal{P}(\tilde\rho+\Lambda_0,8)=\{&(0,0,0,0)_0\ ,\ (2,0,0,2)_2\ ,\ (3,0,1,1)_3\ ,\ (1,1,0,3)_3\ ,\ (4,1,0,1)_4\ ,\\ &(2,1,1,2)_4\ ,\ (1,0,1,4)_4\ ,\ (6,0,0,1)_5\ ,\ (3,2,0,2)_5\ ,\ (2,0,2,3)_5\ ,\\ &(1,0,0,6)_5\ ,\ (0,3,3,0)_6\ ,\ (1,4,2,0)_7\ ,\ (0,2,4,1)_7\ ,\ (2,0,0,7)_7\ ,\\ &(3,0,2,4)_7\ ,\ (4,2,0,3)_7\ ,\ (7,0,0,2)_7\ ,\ (0,6,1,0)_8\ ,\ (0,1,6,0)_8\ ,\\ &(3,0,1,6)_8\ ,\ (4,1,1,4)_8\ ,\ (6,1,0,3)_8\}\end{aligned}$$

where the notation

$$k\ \Lambda_0-M\ \delta+\rho+\sum_{i=1}^{4}\ n_i\mu_i\equiv(n_1,n_2,n_3,n_4)_M$$

is used for convenience. Since by definition

$$\mathcal{P}(\Lambda^+,n)\supset\mathcal{P}(\Lambda^+,m)\quad,\quad n>m\quad,$$

this means in effect that

$$\mathcal{P}(\Lambda^+,8)\supset\mathcal{P}(\Lambda^+,7)\supset\mathcal{P}(\Lambda^+,2)\quad.$$

Now if one conveniently defines

$$\chi_{\Lambda_0}(\tau,M) \equiv \frac{\sum_{\mu^+ \in \mathcal{P}(\tilde{\rho}+\Lambda_0,M)} \; S_{\{\mu^+\}}(x_1,x_2\ldots x_5)}{\sum_{\mu^+ \in \mathcal{P}(\tilde{\rho},M)} \; S_{\{\mu^+\}}(x_1,x_2\ldots x_5)} \quad . \qquad (IV.18)$$

then one sees that $\chi_{\Lambda_0}(\tau,2)$ and $\chi_{\Lambda_0}(\tau,7)$ give respectively the terms explicitly shown in (IV.16) and (IV.17) while $\chi_{\Lambda_0}(\tau,8)$ reproduces (IV.14) up to nineth order.

IV. CONCLUSION

We extend the concept of permutation weights, which we have introduced for finite Lie algebras previously, to affine Lie algebras. This brings us out the formula (III.11) which permits a reformulation of Weyl-Kac character formula. In comparison of (III.11) with (IV.9), our emphasis is that the sum over Weyl groups is removed in the former whereas explicit calculation of permutation weights is a simple task even for the huge algebra of E_8 Lie group. The hope however is that the permutation weights would have a similar meaning also for hyperbolic Lie algebras.

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